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The structure of neighbor disconnected vertex transitive graphs

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Abstract

We define a graph to be *neighbor disconnected* if the removal of the closed neighborhood of a vertex leaves a disconnected induced subgraph. Our main theorem is that a vertex transitive graph is neighbor disconnected if and only if it is a wreath product of vertex transitive graphs, with the necessary restriction that one factor must be neighbor disconnected whenever the other factor is a clique. Among the applications, we describe all connected neighbor disconnected vertex transitive graphs of degree not exceeding 10, and characterize the generating sets of all neighbor disconnected Cayley graphs. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Suppose, in a network, that the failure of a node causes the failure of all adjacent nodes, leaving the nodes at distance two or greater functional. If a node in such a network fails, under what circumstances will the surviving network be disconnected? This question is almost the same as asking for conditions which guarantee that a graph will have neighbor connectivity one [3–5]. In Section 4, we define a graph to be *neighbor disconnected* if it is possible to delete the closed neighborhood of a vertex and thereby leave a graph that is disconnected. Neighbor disconnectedness implies neighbor connectivity one but not conversely, since a graph has neighbor connectivity one if it is either neighbor disconnected or else is neighbor connected with a survival subgraph that is either empty or a clique.

In general, the subgraph that remains after the closed neighborhood of a vertex is removed depends on which vertex is chosen. In this paper, we restrict our attention to vertex transitive graphs, so that the isomorphism type of the surviving subgraph is

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independent of the closed vertex neighborhood that is deleted. In order to make our results as useful as possible, we consider, whenever we can, that our graphs have a transitive action of a specified subgroup of the full automorphism group, and we phrase our arguments and results relative to that subgroup. All results obtained this way can be immediately applied to the Cayley graph of a group G without having to worry about automorphisms of the Cayley graph that do not come from the regular action of G on the graph.

The first attempt at finding general conditions for a class of graphs to have neighbor connectivity one was carried out for a restricted class of Cayley graphs by Doty, Goldstone, and Suffel [2], hereafter referred to as [DGS]. That paper studied the algebraic properties of generating sets that correspond to neighbor disconnection in certain types of Cayley graphs. This paper focuses on intrinsic structural properties of neighbor disconnected vertex transitive graphs, properties that rely on the existence of a transitive automorphism group but can be described without reference to its nature. This shift makes it possible to treat all vertex transitive graphs, rather than just a special class of Cayley graphs, and still obtain the results of [DGS] without the restrictive hypotheses that paper required.

Although this paper takes a different approach than [DGS], it makes use of a fundamental insight from [DGS], namely, that the structure of vertex transitive neighbor disconnected graphs can be understood by examining the sets of vertices that are invisible from maximum components.

2. Overview

The 13 sections of the paper can be grouped into three parts. The first part, Sections 1–4, contains the introductory material for the rest of the paper. Section 3 collects conventions and notations that are in general use and will be employed in this paper, while Section 4 introduces terms and notations that are specific to this paper and adduces some elementary consequences.

The second and main part of the paper, Sections 5–12, develops the technical machinery that culminates in wreath product structure theorems for neighbor disconnected vertex transitive graphs, and applies those theorems to obtain various specific results about neighbor connectivity. The main development takes place in Sections 7–10. A more detailed overview of that development then we can give at this point, relying on concepts and results introduced in the first five sections, is presented in Section 6. Section 11 illustrates the need for our modification of the original notion of 'neighbor connectivity one' in the light of the structure theorems, and Section 12 contains the applications.

The third part of the paper, Section 13, develops necessary and sufficient conditions for neighbor disconnection in terms of neighborhoods and orbits of the vertex whose closed neighborhood is deleted. These theorems offer a different perspective on neighbor disconnection, and are especially suited to Cayley graphs that are defined in terms of generating subsets of the group. The Cayley graph applications finish the work begun in [DGS] on characterizing neighbor disconnection in Cayley graphs in terms of their generating sets.

3. Background

We use the square union symbol '$\sqcup$' to denote the union of two sets and to convey the additional information that they are disjoint.

All graphs considered are finite and undirected. We use K_t, $t \geqslant 1$, to denote the complete graph, or clique, of order t, E_t, $t \geqslant 1$, to denote the empty graph of order t, namely, the complement of K_t, C_t, $t \geqslant 3$, to denote the cycle graph of order t, $K_{t,t,\dots,t}$, $t \geqslant 1$, to denote the complete multipartite graph consisting of copies of E_t completely joined in pairs, and $\varnothing$ to denote the null graph (with empty vertex set). All other graphs will be denoted by capital Greek letters, and we assume that any such graph is non-null. If Γ denotes a graph, then $V\Gamma$ denotes the vertex set of Γ, $E\Gamma$ denotes the edge set of Γ, $\mathscr{C}\Gamma$ denotes the complement of Γ, and $|\Gamma|$ denotes the order of Γ. We use the symbol '$\sim$' to denote the adjacency relation for vertices. If $W \subseteq V\Gamma$, we use $\mathrm{Gr}(W)$ to denote the subgraph of Γ induced by W. If necessary for clarity, we subscript '$\sim$' and 'Gr' with the name of the graph. If W_1 and W_2 are sets of vertices of Γ, we refer to them as *completely joined* if every pair of vertices in $W_1 \times W_2$ is adjacent. We refer to W_1 and W_2 as *isolated from each other* if no pair of vertices in $W_1 \times W_2$ is adjacent. (Our formulation admits the useful peculiarity that any subset of vertices is both completely joined to and isolated from the empty set.) If Γ is a graph and $\mathscr{B} = \{B_i\}$ is a partition of $V\Gamma$, then by $\Gamma_{\mathscr{B}}$ we mean the graph whose vertices are the subsets $B_i \in \mathscr{B}$ and whose adjacency relation is $B_i \sim B_j$ in $\Gamma_{\mathscr{B}}$ if and only if there is an $x \in B_i$ and a $y \in B_j$ with $x \sim y$ in Γ.

If the graph Γ admits a left action of a group G, we refer to Γ as a *G-graph*. All actions will be written multiplicatively. If the action of G is transitive on $V\Gamma$, meaning that for any $x, y \in V\Gamma$, there is an automorphism of Γ carrying x to y, then Γ is called a *vertex transitive* G-graph. If the action of G preserves no non-trivial partitions of the elements of $V\Gamma$, then the action is said to be *primitive*. Otherwise, the action is *imprimitive*, and any non-trivial partition preserved by the G-action is called a *block system* for the action of G. If the action of G on $V\Gamma$ is regular, meaning that G acts transitively on $V\Gamma$ and no vertex of Γ is fixed by any non-identity group element, then Γ is said to be a *Cayley graph* of the group G. An equivalent definition of (undirected) Cayley graphs associates a graph, denoted in this paper by $\mathrm{Cay}(G,S)$, with every pair (G,S) in which G is a group and S is a subset of G that is closed under inverses and does not contain the identity element. By definition, $V\,\mathrm{Cay}(G,S) = G$ and $E\,\mathrm{Cay}(G,S) = \{(g,gs) \mid g \in G, s \in S\}$, and it is easy to see that $\mathrm{Cay}(G,S)$ is connected if and only if S generates G. For more details on vertex transitive graphs and Cayley graphs, the reader may consult the relevant parts of Chapters 16 and 22 of [1].

It is an unfortunate fact of life that 'regular' is an overused word in mathematics. In this paper, two usages occur. When we speak of a *subgroup* of $\mathrm{Aut}(\Gamma)$ or an *action* as being regular, we are using regularity in the sense defined in the previous paragraph. When we speak of a *graph* as being regular, we mean that all vertices have the same degree. Vertex transitive graphs are regular in the second sense but not necessarily in the first sense. The common degree of the vertices of a regular graph Γ will be denoted $\deg(\Gamma)$.

For $W \subseteq V\Gamma$, we use G_W to denote the *setwise stabilizer of* W, i.e. $G_W = \{g \in G \mid gw \in W \text{ for all } w \in W\}$. There does not seem to be a universally accepted notation for the setwise stabilizer, and the symbol G_W is often used to denote the *pointwise stabilizer of* W, namely the set $\{g \in G \mid gw = w \text{ for all } w \in W\}$. Since we never have occasion to refer to the pointwise stabilizer of a set with more than one element, no confusion should arise from this choice of notation.

Let Σ and Δ be graphs. The *wreath product* of Σ and Δ, denoted by $\Sigma \wr \Delta$, is central to our structural characterization of vertex transitive graphs with disconnected survival subgraphs. The wreath product is also called the *composition product* and the *lexicographic product*, and is often denoted by $\Sigma[\Delta]$. We have chosen not to use this notation because it makes the representation of iterated wreath products cumbersome, forcing us to choose some particular pattern of association even though the wreath product is associative.

The definition of the wreath product of Σ and Δ is as follows: $V(\Sigma \wr \Delta) = V\Sigma \times V\Delta$ and for $x, x' \in V\Sigma$ and $y, y' \in V\Delta$,

$$(x', y') \underset{\Sigma \wr \Delta}{\sim} (x, y) \Leftrightarrow \begin{cases} x' \underset{\Sigma}{\sim} x, & \text{or} \\ x' = x & \text{and } y' \underset{\Delta}{\sim} y. \end{cases}$$

Routine checks show that the wreath product is associative up to isomorphism, that $\mathscr{C}(\Sigma \wr \Delta) = \mathscr{C}\Sigma \wr \mathscr{C}\Delta$, and that if Σ and Δ are regular graphs, then $\deg(\Sigma \wr \Delta) = \deg(\Sigma)|\Delta| + \deg(\Delta)$.

In the wreath product $\Sigma \wr \Delta$, we refer to Σ as the *base* of the wreath product and Δ as the *fiber*. A wreath product is *non-trivial* if neither the base nor the fiber consists of a single vertex.

Wreath products of graphs are related to wreath products of permutation groups. In particular, if Σ is a vertex transitive G-graph and Δ is a vertex transitive H-graph, then $\Sigma \wr \Delta$ is a vertex transitive $G \wr H$-graph. $G \wr H$ is a semidirect product that may be described as follows: let $V\Sigma = \{s_1, s_2, \ldots, s_m\}$ and $V\Delta = \{d_1, d_2, \ldots, d_n\}$. Then we may view the elements of $G \wr H$ as tuples $(g, h_{s_1}, h_{s_2}, \ldots, h_{s_m})$ with $g \in G$ and $h_{s_i} \in H$, so the order of $G \wr H$ is $|G||H|^{|V\Sigma|}$. The action of such a tuple on a typical vertex (s_i, d_j) of $\Sigma \wr \Delta$ is $(g, h_{s_1}, h_{s_2}, \ldots, h_{s_m})(s_i, d_j) = (gs_i, h_{s_i}d_j)$. If the G and H actions are regular, so that Σ and Δ are Cayley graphs, then $G \wr H$ contains a regular subgroup consisting of all elements of the form $(g, h, h, \ldots, h)$, $g \in G$, $h \in H$, and so $\Sigma \wr \Delta$ is a Cayley graph.

The following proposition collects some additional well-known facts about wreath products of graphs and their automorphism groups.

Proposition 1. *Let Γ be a graph whose vertex set is equipped with a partition $\mathscr{B} = \{B_i\}$. Suppose that the subgraphs induced by each B_i are isomorphic to a fixed graph with vertex set B, and that every distinct pair of subsets B_i, B_j are either isolated from each other or are completely joined in Γ. Then*

1. $\Gamma \cong \Gamma_{\mathscr{B}} \wr \mathrm{Gr}(B)$;
2. $\mathrm{Aut}(\Gamma_{\mathscr{B}}) \wr \mathrm{Aut}(\mathrm{Gr}(B))$ *is isomorphic to a subgroup of* $\mathrm{Aut}(\Gamma)$;
3. *If G is any group of automorphisms of Γ that preserves $\mathscr{B}$, i.e. $gB_i \in \mathscr{B}$ for all i and all $g \in G$, then G is isomorphic to a subgroup of* $\mathrm{Aut}(\Gamma_{\mathscr{B}}) \wr \mathrm{Aut}(\mathrm{Gr}(B))$.

Definition 2. For $W \subseteq V\Gamma$. Let $A(W) = \{v \in V\Gamma \mid v \sim w$ for some $w \in W\}$. Set

$$N(W) = A(W) \backslash W; \qquad N[W] = A(W) \cup W.$$

$N(W)$ is called the *open neighborhood* of W and $N[W]$ the *closed neighborhood* of W. If necessary for clarity, we write $N_\Gamma(W)$ and $N_\Gamma[W]$.

Some authors refer to $N(W)$ as the *boundary* of W and use the notation ∂W. We find it most useful to think of $N(W)$ as the set of vertices outside W that are *visible* from W.

4. Preliminaries

Definition 3. Let G be a group, V a G-set, $x \in V$, and $W \subseteq V$. Define

$$G(x, W) = \{g \in G \mid gx \in W\}.$$

A fundamental fact about groups acting on sets is that if G acts transitively on V, then $G(x, W)$ is a union of cosets of G_x, and these cosets are in 1–1 correspondence with the elements of W.

Definition 4. For $W \subseteq V\Gamma$, define $I(W) = V\Gamma \backslash N[W]$. We call $I(W)$ the set of vertices *invisible from W*.

We shall have occasion to use the following facts:

Lemma 5. *Let U and W be subsets of $V\Gamma$ and x an element of $V\Gamma$. Then*

1. *If $U \subseteq W$, then $I(U) \supseteq I(W)$*;
2. $W \subseteq I^2(W)$;
3. $V\Gamma = W \sqcup N(W) \sqcup I(W)$;
4. $N_{\mathscr{C}\Gamma}(x) = I_\Gamma(x)$;
5. $I_{\mathscr{C}\Gamma}(x) = N_\Gamma(x)$.

Proof. Immediate from the definitions. □

Using the interpretation of $N(W)$ as the set of vertices outside W that are visible from W, Part 3 of Lemma 5 says that the vertices of Γ are the disjoint union of W, the vertices outside W that are visible from W, and the vertices outside W that are invisible from W.

Definition 6. For any vertex $x \in V\Gamma$, let $S_x\Gamma = \mathrm{Gr}(I(x))$. We call $S_x\Gamma$ the *x-survival subgraph of* Γ and refer to x as the *center of deletion.*

Gunther and Hartnell's definition of neighbor connectivity allows for more general types of survival subgraphs than we consider here. When we speak of a *survival subgraph* of Γ, we always mean an x-survival subgraph for some $x \in V\Gamma$. If Γ is vertex transitive, then the isomorphism type of a survival subgraph of Γ is independent of the center of deletion. (See Corollary 14.) Thus, for vertex transitive graphs Γ, we will use the symbol $S\Gamma$ to denote a survival subgraph when we do not need to focus on the center of deletion.

Note that it is possible that $S_x\Gamma$ is the null graph. For vertex transitive graphs, $S\Gamma = \varnothing$ if and only if Γ is a clique.

We shall frequently consider disjoint unions of iterated wreath products in which some factors may be survival subgraphs of a graph. In order to avoid a profusion of parentheses, we employ the following association conventions:

Convention. In the absence of parentheses, when the survival operator S appears in an iterated wreath product, it only applies to the factor immediately to its right, and in an expression containing both wreath products and disjoint unions, all wreath products are carried out before disjoint unions.

According to this convention, the expression $S_x\Sigma \wr \Delta \sqcup S_y\Delta$ denotes $((S_x\Sigma) \wr \Delta) \sqcup S_y\Delta$.

Definition 7. Let Γ be a graph. We say Γ is *neighbor disconnected* if there exists a vertex $v \in V\Gamma$ such that $S_v\Gamma$ is disconnected. If no such vertex exists, then Γ is *neighbor connected.*

We consider the null graph to be connected. Consequently, according to our definition, cliques are neighbor connected.

Lemma 8. *Let* $\Gamma = \Sigma \wr \Delta$. *Then* $S_{(x,y)}\Gamma \cong S_x\Sigma \wr \Delta \sqcup S_y\Delta$.

Proof. To verify the isomorphism of the lemma, negate the definition of the adjacency relations for $\Sigma \wr \Delta$ and rearrange the clauses obtained into a mutually exclusive pair. The result is

$$(x',y') \underset{\Gamma}{\not\sim} (x,y) \Leftrightarrow \begin{cases} x' \underset{\Sigma}{\not\sim} x \text{ and } x' \neq x, \text{ or} \\ x' = x \text{ and } y' \underset{\Delta}{\not\sim} y. \end{cases}$$

The first condition is equivalent to $(x', y') \in S_x\Sigma \wr \Delta$, the second condition is equivalent to $(x', y') \in S_{(x,y)}(x \times \Delta)$, and $S_{(x,y)}(x \times \Delta) \cong S_y\Delta$. $\square$

Corollary 9. *$\Sigma \wr \Delta$ is neighbor disconnected unless one of Σ and Δ is neighbor connected and the other of Σ and Δ is a clique.*

Proof. The only way $S_x\Sigma \wr \Delta \sqcup S_y\Delta$ can be connected is if one of the two summands of the disjoint union is connected and the other is empty. $S_x\Sigma \wr \Delta$ is empty if and only if Σ is a clique, and $S_y\Delta$ is empty if and only if Δ is a clique. $\square$

Proposition 10. *Let $\Gamma = \Sigma_1 \wr \Sigma_2 \wr \cdots \wr \Sigma_m$ $(m \geqslant 2)$. Then*

$$S_{(x_1,x_2,\ldots,x_m)}\Gamma \cong S_{x_1}\Sigma_1 \wr \Sigma_2 \wr \cdots \wr \Sigma_m \sqcup S_{x_2}\Sigma_2 \wr \Sigma_3 \wr \cdots \wr \Sigma_m \sqcup \cdots \sqcup S_{x_{m-1}}\Sigma_{m-1} \wr \Sigma_m \sqcup S_{x_m}\Sigma_m.$$

Proof. A routine induction on m using Lemma 8. $\square$

Definition 11. Let Γ be a graph and $x \in V\Gamma$. U_x will always denote the union of the vertices of the maximum components of $S_x\Gamma$.

The disjoint union $V\Gamma = U_x \sqcup N(U_x) \sqcup I(U_x)$ is at the core of our analysis of the structure of Γ when Γ is vertex transitive and $S_x\Gamma$ is disconnected. Note that $I(U_x)$ is never empty, since it always contains at least x. For vertex transitive Γ, $U_x = \emptyset$ if and only if Γ is a clique, and $N(U_x) = \emptyset$ if and only if Γ is either a clique or is disconnected.

Proposition 12. *Let Γ be a G-graph. Then $A(\cdots)$, $N(\cdots)$, $N[\cdots]$, and $I(\cdots)$ are all G-maps of the power set of $V\Gamma$.*

Proof. If $F(\cdots)$ denotes any one of these operators, then the proposition asserts that for any $W \subseteq V\Gamma$ and any $g \in G$, $F(gW) = gF(W)$. This is easily checked when $F = A$, and the results for the remaining operators follow from the G-equivariance of A and the fact that the G-action distributes over the set operations $\cup$ and $\setminus$. $\square$

Lemma 13. *Let Γ be a G-graph. Any $g \in G$ induces an isomorphism $g : S_x\Gamma \to S_{gx}\Gamma$.*

Proof. Any such g preserves adjacency, and $VS_{gx}\Gamma = I(gx) = gI(x) = g(VS_x\Gamma)$. $\square$

Corollary 14. *If Γ is a vertex transitive graph, then the isomorphism type of $S_x\Gamma$ is independent of the choice of $x \in V\Gamma$.*

Lemma 15. *Let Γ be a G-graph. Then for every $g \in G$, $gU_x = U_{gx}$, $gN(U_x) = N(U_{gx})$, and $gI(U_x) = I(U_{gx})$.*

Proof. Each $g \in G$ induces a graph isomorphism $g : S_x\Gamma \to S_{gx}\Gamma$ by Lemma 13. Such an isomorphism must carry maximum components isomorphically onto maximum components, so $gU_x = U_{gx}$. Apply $N(\cdots)$ and $I(\cdots)$ to both sides of this equation and factor g out of the left hand side to obtain the other conclusions of the lemma. $\square$

5. Graphs with empty survival subgraphs

It is so easy to characterize vertex transitive graphs whose survival subgraphs are empty graphs that it is worthwhile to do so before embarking on the involved proof for the general case. We first record an elementary fact that has been promoted to the status of a lemma because it is referred to many times in subsequent arguments.

Lemma 16. *Γ is a vertex transitive graph if and only if $\Gamma \cong E_t \wr \Delta$, where Δ is a connected vertex transitive graph and $t \geqslant 1$ is the number of components of Γ.*

Proof. If Γ is a vertex transitive graph, then it is a disjoint union of isomorphic vertex transitive components. In other words, $\Gamma \cong E_t \wr \Delta$ with Δ a connected vertex transitive graph and $t \geqslant 1$ the number of components of Γ.

Conversely, if $\Gamma \cong E_t \wr \Delta$ with Δ a vertex transitive H-graph, then the group $S_t \wr H$ acts transitively on $V\Gamma$ and so Γ is a vertex transitive graph which is disconnected if $t \geqslant 2$. This proves the lemma. $\square$

The reason that Lemma 16 is important for vertex transitive graphs with empty survival subgraphs is that the complements of such graphs are disconnected vertex transitive graphs, a fact that is exploited in the proof of the main theorem of this section.

Theorem 17. *Γ is a vertex transitive graph with empty survival subgraph E_t, $t \geqslant 1$, if and only if $\Gamma \cong K_s \wr E_{t+1}$ for some $s \geqslant 1$.*

Proof. Suppose $S_x\Gamma \cong E_t$. Using Lemma 5, Part 4, we have $\mathrm{Gr}_{\mathscr{C}\Gamma} N_{\mathscr{C}\Gamma}(x) = \mathrm{Gr}_{\mathscr{C}\Gamma} I_\Gamma(x) = \mathscr{C}\mathrm{Gr}_\Gamma I_\Gamma(x) = \mathscr{C}E_t = K_t$. Hence, $\mathrm{Gr}_{\mathscr{C}\Gamma} N_{\mathscr{C}\Gamma}[x] = K_{t+1}$, which means, in view of the regularity of Γ, that $N_{\mathscr{C}\Gamma}[x]$ is the vertex set of a component of $\mathscr{C}\Gamma$. By Lemma 16, $\mathscr{C}\Gamma \cong E_s \wr K_{t+1}$ for some $s \geqslant 1$. Taking the complement of $\mathscr{C}\Gamma$, we get $\Gamma \cong K_s \wr E_{t+1}$. Conversely, if $\Gamma \cong K_s \wr E_{t+1}$, then by Lemma 8, $S_{(x,y)}\Gamma \cong S_y E_{t+1} \cong E_t$ as required. $\square$

6. Overview of the proof for the general case

When the survival subgraph of Γ is disconnected but not an empty graph, we have to work harder to uncover the wreath product structure of Γ. Our goal is to use the fact that $S_x\Gamma$ is disconnected to find a partition $\mathscr{B}$ of $V\Gamma$ into the fiber sets of a non-trivial wreath product with vertex transitive factors. Thus, any two fiber sets in the partition must have isomorphic vertex transitive induced graphs and must be either isolated from each other or completely joined.

Our goal is achieved in Sections 7–10 by showing that if U_x, the union of maximum components of $S_x\Gamma$, consists of a single maximum component, then the desired fiber sets have the form $I(U_x)$ as x runs through a representative set of vertices of $V\Gamma$. If U_x contains more than one maximum component, then the fiber sets have the form $U_x \sqcup I(U_x)$ as x runs through a representative set of vertices of $V\Gamma$.

In Section 7, we begin this program by working out the relationship between the setwise stabilizers of x, $I(U_x)$, U_x, and $N(U_x)$ and establishing the transitivity of the action of $G_{I(U_x)}$ on $I(U_x)$, one of the requirements for the type of wreath product decomposition we seek. Lemmas 18 and 19 are among the core technical results of the paper.

In Section 8, we study some completely joined subsets of vertex transitive graphs. The results are later used to show that partitions using the sets $I(U_x)$ and $U_x \sqcup I(U_x)$ consist of sets that are either isolated from each other or are completely joined, as is required in order for these partitions to induce wreath product structures. Lemma 21 in this section is another of the core technical results of the paper.

In Section 9, we lay the groundwork for the proof that partitions using subsets of the form $I(U_x)$ (in case there is a single maximum component of the survival subgraph) and partitions using subsets of the form $U_x \sqcup I(U_x)$ (in case there are multiple maximum components of the survival subgraph) induce wreath product decompositions. Much of the analysis of these partitions is the same, so to avoid a great deal of repetition, we abstract the common features of the arguments for the two partitions by defining the concept of *wreath product inducers* and prove, in Lemma 26, that wreath product inducers do indeed correspond to wreath product decompositions of our graphs.

In Section 10, the work of Section 9 culminates in the proof of the main theorems of the paper. The arguments for the two types of partitions are carried out in Theorems 29 and 30, which invoke the wreath product inducer machinery of the previous section. The two theorems are then combined to yield the most general results of the paper, Theorems 31 and 33, which give necessary and sufficient conditions for neighbor disconnection, the second theorem in terms of a wreath product decomposition into 'indecomposable factors'. We also establish an interesting result, Theorem 32, indicating that the wreath product decompositions we obtain are adapted to the calculation of the automorphism group of our graph, a property that Sabadussi has shown in [6] to be unavailable for arbitrary wreath product decompositions.

7. Subsets and stabilizers of G-graphs

Lemma 18. *Let Γ be a G-graph with $x \in V\Gamma$. Then $G_{I(U_x)} = G(x, I(U_x))$.*

Proof. Streamline the notation by letting $U = U_x$. Since $x \in I(U)$, it is immediate that $G_{I(U)} \subseteq G(x, I(U))$. The opposite inclusion is equivalent to the statement that

$G(x,I(U))I(U)=I(U)$, and since $I(\cdots)$ is a G-map, the statement $G(x,I(U))U=U$ is sufficient to establish the opposite inclusion. Thus, it suffices to show that if $g\in G(x,I(U))$, then $gU=U$. We have equivalences

$$\begin{aligned} g\in G(x,I(U)) &\Leftrightarrow \{gx\}\subseteq I(U), \\ &\Leftrightarrow I(gx)\supseteq I^2(U)\supseteq U, \quad \text{by Lemma 5, parts 1 and 2,} \\ &\Leftrightarrow I(x)\supseteq g^{-1}U, \quad \text{using Proposition 12.} \end{aligned}$$

The last line above says that $VS_x\Gamma$ contains $g^{-1}U$. In other words, $S_x\Gamma$ contains the subgraph $\mathrm{Gr}(g^{-1}U)$, which is isomorphic to $\mathrm{Gr}(U)$. Because of this isomorphism, each component of $\mathrm{Gr}(g^{-1}U)$ is a connected subset of $S_x\Gamma$ of maximum order, and so must be a maximum component of $S_x\Gamma$. Hence $g^{-1}U=U$, or equivalently, $gU=U$, the result needed to conclude the proof. □

Lemma 19. *Let Γ be a vertex transitive G-graph and let $x\in V\Gamma$. Then*
1. *$G_{I(U_x)}$ acts transitively on $I(U_x)$;*
2. *$G_x\leqslant G_{I(U_x)}=G_{U_x}\leqslant G_{N(U_x)}$.*

Proof. We continue using the streamlined notation $U=U_x$. Since G is vertex transitive, every element of $I(U)$ has the form gx for some $g\in G(x,I(U))$, so Lemma 18 shows that $G_{I(U)}$ is transitive on $I(U)$. This proves Part (1) of the lemma, and the statement $G_x\leqslant G_{I(U_x)}$ in Part (2) follows immediately from Lemma 18 and the fact that $G_x\subseteq G(x,I(U))$.

To prove the remaining pair of facts in Part (2) of the lemma, recall from the proof of Lemma 18 that $G(x,I(U))U=U$, which becomes $G_{I(U)}U=U$ in light of the conclusion of Lemma 18. Thus, $G_{I(U)}\leqslant G_U$. Since $N(\cdots)$ and $I(\cdots)$ are G-maps, it follows that $G_UI(U)=I(U)$ and $G_UN(U)=N(U)$, whence $G_U\leqslant G_{I(U)}$ and $G_U\leqslant G_{N(U)}$. This concludes the proof. □

The inclusions in Part (2) of Lemma 19 can be proper. In view of the fact that $G_{I(U_x)}$ acts transitively on $I(U_x)$, the first inclusion will be proper for any vertex transitive graph in which $I(U_x)$ contains more than the vertex x. For example, take $\Gamma=C_m\wr E_2$ for $m\geqslant 5, G=\mathrm{Aut}(\Gamma)$. For these graphs, $I(U_x)=\{x,y\}$, where y is the other vertex in the copy of E_2 containing x.

An example in which the second inclusion is proper is $\Gamma=C_4$, $G=\mathrm{Aut}(\Gamma)$. Let $VC_4=\{1,2,3,4\}$. Then $S_1C_4=\{3\}=U_1$, $N(U_1)=\{2,4\}$, $G_{U_1}=\{e,(2\,4)\}$, and $G_{N(U_1)}=\{e,(1\,3),(2\,4),(1\,3)(2\,4)\}$.

8. Complete joins of subsets

Lemma 20. *Let Γ be a vertex transitive G-graph and let $x\in V\Gamma$. Then $I(U_x)$ is completely joined to $N(U_x)$ in Γ.*

Proof. If either U_x or $N(U_x)$ is empty, the conclusion is vacuously true. Otherwise, by definition, each vertex of $N(U_x)$ is adjacent to a vertex of a maximum component of $S_x\Gamma$. Hence, no vertex of $N(U_x)$ can be in $S_x\Gamma$, else some maximum component would be larger. Consequently, x must be completely joined to $N(U_x)$, and so the orbit of x under any subgroup of $G_{N(U_x)}$ is completely joined to $N(U_x)$. According to Lemma 19, $G_{I(U_x)}$ is a subgroup of $G_{N(U_x)}$ that is transitive on $I(U_x)$, so $I(U_x)$ must be completely joined to $N(U_x)$ as claimed. $\square$

Lemma 21. *Let Γ be vertex transitive G-graph and $x \in V\Gamma$. Suppose that $S_x\Gamma$ is disconnected with at least two maximum components. Then U_x is completely joined to $N(U_x)$.*

Proof. U_x is non-empty by hypotheses. If $N(U_x)$ is empty, the conclusion is vacuously true. Otherwise, Let $C \subseteq U_x$ denote the vertex set of one of the maximum components of $S_x\Gamma$. Suppose $c \in C$ and $y \in N(U_x)$ are not adjacent. By definition of $N(U_x)$, y is adjacent to a vertex of some maximum component of $S_x\Gamma$. There are two cases to consider:

Case 1: $y \sim d \in D$, where D is the vertex set of a maximum component distinct from C. In this case, $D \cup \{y\}$ is a connected subset of $S_c\Gamma$, which is impossible since all survival subgraphs of Γ are isomorphic and the cardinality of D is already the maximal cardinality for connected subsets of any survival subgraph of Γ.

Case 2: $y \sim c' \in C$. There is, by hypothesis, another maximum component with vertex set D. Let $d \in D$ be any vertex. By the argument in Case 1, $d \sim y$, since otherwise $C \cup \{y\}$ will be an impermissibly large connected subset of $S_d\Gamma$. This reduces Case 2 to Case 1, which has already been shown to yield a contradiction.

Thus $c \sim y$ in all cases and so the lemma is proved. $\square$

Lemma 22. *Let Γ be a vertex transitive G-graph and $x \in V\Gamma$. Then the following conditions are equivalent*:

1. *U_x is completely joined to $N(U_x)$.*
2. *For each vertex $y \in U_x$, the maximum component of $S_x\Gamma$ containing y is* $\mathrm{Gr}(I(U_y))$.
3. *$G_{N(U_x)}$ acts transitively on $U_x \sqcup I(U_x)$.*

Furthermore, if any of the above equivalent conditions hold and if the survival subgraph has no submaximum components, then each component of the survival subgraph is a clique K_m for some fixed integer $m \geq 1$.

Proof. If U_x is empty, i.e. if Γ is a clique, then the three statements are vacuously true and so are equivalent. For the rest of the proof, we assume $U_x \neq \emptyset$.

$1 \Rightarrow 2$: We begin by proving that $\mathrm{Gr}(I(U_a))$ is a maximum component of $S_b\Gamma$ for any $b \in U_a$. Let $C_1, C_2, \ldots, C_t$, $t \geq 1$, be the vertex sets of the maximum components of $S_a\Gamma$, so that $U_a = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_t$. If $b \in U_a$, then $b \in C_i$ for some i, $1 \leq i \leq t$. The vertices in C_j for $j \neq i$ (if there are any) and the vertices in $I(U_a)$ are all isolated from C_i in Γ and, by assumption, $N(U_a)$ is completely joined to C_i in Γ. Therefore,

$$S_b\Gamma = \mathrm{Gr}(C_i\backslash N[b]) \sqcup \mathrm{Gr}(I(U_a)) \sqcup \bigsqcup_{j\neq i} \mathrm{Gr}(C_j). \tag{1}$$

(If $t=1$, then the part of the disjoint union above indexed by $j \neq i$ is empty.) $C_i\backslash N[b]$ is either empty (if $\mathrm{Gr}(C_i)$ is a clique) or else is the vertex set of one or more submaximum components of $S_b\Gamma$. $S_b\Gamma$ has to have t maximum components, so the required additional component isomorphic to $\mathrm{Gr}(C_i)$ must be a component of $\mathrm{Gr}(I(U_a))$. According to Lemma 19, $\mathrm{Gr}(I(U_a))$ is a vertex transitive graph, so if it is disconnected, it would have to be a union of components isomorphic to $\mathrm{Gr}(C_i)$. But if $\mathrm{Gr}(I(U_a))$ consists of two or more isomorphic copies of $\mathrm{Gr}(C_i)$, then $S_b\Gamma$ would have more than t maximum components, which cannot be. Thus, $\mathrm{Gr}(I(U_a))$ is a maximum component of $S_b\Gamma$ for any $b\in U_a$ as claimed.

Taking $a=y$ and $b=x$ in the previous result, we find that $\mathrm{Gr}(I(U_y))$ is a maximum component of $S_x\Gamma$ for any $x\in U_y$. But what we want to prove is that $\mathrm{Gr}(I(U_y))$ is a maximum component of $S_x\Gamma$ for any $y\in U_x$, so we must show that $x\in U_y$ if and only if $y\in U_x$.

Suppose, then, that $x\in U_y$. Then $I(U_y)$, as the vertex set of a maximum component of $S_x\Gamma$, must be contained in U_x. Hence $y\in I(U_y)\subseteq U_x$. For the opposite inclusion, suppose that $y\in U_x$. In order to argue as before that $x\in I(U_x)\subseteq U_y$, we need to know that $I(U_x)$ is the vertex set of a maximum component of $S_y\Gamma$ whenever $y\in U_x$. This fact emerges from our previous result if we take $a=x$ and $b=y$, and so the implication $1\Rightarrow 2$ is proved.

$2\Rightarrow 3$: We shall prove that $G_{N(U_x)}x = I(U_x)\sqcup U_x$. This is immediate if $N(U_x)=\emptyset$, so suppose $N(U_x)\neq\emptyset$. Since $S_x\Gamma = N(U_x)\sqcup I(U_x)\sqcup U_x$, the inclusion $G_{N(U_x)}x\subseteq I(U_x)\sqcup U_x$ follows immediately from $x\in I(U_x)$. For the opposite inclusion, note first, using Lemma 19, that $I(U_x)=G_{I(U_x)}x\subseteq G_{N(U_x)}x$. It remains to be seen that $U_x\subseteq G_{N(U_x)}x$. Because of the transitivity of G on $V\Gamma$, we can let gx denote an arbitrary element of U_x. We then show that $g\in G_{N(U_x)}$ as follows: By definition of U_x, $gx\in C$, where C is the vertex set of a maximum component of $S_x\Gamma$. Let $y\in N(U_x)$. We must show that $gy\in N(U_x)$. By Lemma 20, $I(U_x)$ is completely joined to $N(U_x)$. Thus x and y are adjacent, so gx and gy must be adjacent, which means that gy is either in $N(U_x)$ as we would like to conclude, or else $gy\in C$. If $gy\in C$, then by Condition 2 of this lemma and Lemma 15, $C=I(U_{gy})=gI(U_y)$. Since we know $gx\in C$, the same reasoning shows $C=gI(U_x)$, and thus $I(U_y)=I(U_x)$, forcing $y\in I(U_x)$. But y was chosen to be in $N(U_x)$, which is disjoint from $I(U_x)$. This contradiction establishes the alternative possibility $gy\in N(U_x)$, and so completes the verification of $2\Rightarrow 3$.

$3\Rightarrow 1$: This is immediate if $N(U_x)=\emptyset$, so suppose $N(U_x)\neq\emptyset$. Since $x\in I(U_x)$, x is completely joined to $N(U_x)$ by Lemma 20. Therefore, $G_{N(U_x)}x$ is completely joined to $N(U_x)$, and $G_{N(U_x)}x = U_x\sqcup I(U_x)$ by assumption. Thus U_x is completely joined to $N(U_x)$, as was to be proved.

Finally, we prove that if Condition 1 of this lemma holds and if there are no submaximum components, then each component is a clique K_m for some fixed integer m.

This is an immediate consequence of Equation (1), because if $S_b\Gamma$ has a component C_i that is not a clique, then $C_i \backslash N[b]$ will consist of the vertices of one or more submaximum components of $S_b\Gamma$, and we are assuming in this part of the lemma that no submaximum components exist. □

9. Wreath product inducers

The definitions in this section are adapted to connected vertex transitive graphs, which have $N(U_x) \neq \emptyset$. This is all we need to handle both the connnected and the disconnected case.

Definition 23. Let Γ be a vertex transitive G-graph. Suppose that for each $x \in V\Gamma$, there is a partition $V\Gamma = M_x \sqcup B_x \sqcup N_x$, with B_x and N_x non-empty, such that

1. B_x is isolated from M_x and completely joined to N_x;
2. $x \in B_x$;
3. $B_{gx} = gB_x$ for all $g \in G$;
4. A subgroup of G_{N_x} acts transitively on B_x.

Then the set of all partitions $\{M_x \sqcup B_x \sqcup N_x \mid x \in V\Gamma\}$ is called a *wreath product inducer.*

Lemma 24. *Let Γ be a vertex transitive G-graph, let $\{M_x \sqcup B_x \sqcup N_x \mid x \in V\Gamma\}$ be a wreath product inducer, let $\mathscr{A} = \{B_x \mid x \in V\Gamma\}$, and let $\mathscr{B}$ be a complete set of distinct elements of $\mathscr{A}$. Then $\mathscr{B}$ is a block system for the action of G on $V\Gamma$.*

Proof. $\mathscr{B}$ is a G-equivariant collection of subsets of $V\Gamma$ by Part 3 of Definition 23. The elements of $\mathscr{B}$ cover $V\Gamma$ by Part 2 of the definition. To show that any two elements of $\mathscr{B}$ are disjoint subsets of $V\Gamma$, we prove that any two elements B_x, $B_y \in \mathscr{A}$ are either disjoint or identical. For this, it suffices to prove $B_x = B_y$ if and only if $y \in B_x$. If $B_x = B_y$, then $y \in B_x$ by Part 2 of the definition. Conversely, if $y \in B_x$, then by Part 4 of the definition, we can write $y = gx$ for some $g \in G_{N_x}$ such that $gB_x = B_x$. We then have $B_y = B_{gx} = gB_x = B_x$, the second equality by Part 3 of the definition. Thus, $\mathscr{B}$ is a G-equivariant partition of $V\Gamma$. Since G acts transitively on $V\Gamma$, $\mathscr{B}$ is a block system for this action. □

Lemma 25. *Let Γ be a vertex transitive G-graph with a wreath product inducer as in Lemma* 24, *and let $\mathscr{B}$ be as in Lemma* 24. *Then for each $x \in V\Gamma$, $\mathscr{B}$ is a subpartition of $M_x \sqcup B_x \sqcup N_x$ with at least two elements, and distinct blocks $B_x, B_y \in B$ are either isolated from each other or are completely joined.*

Proof. First, we show that distinct blocks B_x and B_y of vertices of Γ are either isolated from each other or completely joined. To prove this, suppose that B_x and B_y are distinct blocks that are not isolated from each other, so that $B_x \cap B_y = \emptyset$ and there is an $a \in B_x$ adjacent to a $b \in B_y$. The proof of Lemma 24 indicates that $B_x = B_a$ and $B_y = B_b$.

Consider which summand of the disjoint union $V\Gamma = B_a \sqcup M_a \sqcup N_a$ contains b. Since b is adjacent to a, b cannot lie in M_a by Part 1 of Definition 23. Since $B_a \cap B_b = \emptyset$ and $b \in B_b$, b cannot lie in B_a. Hence b must lie in N_a, which is completely joined to B_a by Part 1 of Definition 23. Thus, $B_a \subseteq N(b)$. But $N(b) \subseteq B_b \sqcup N_b$ by Parts 1 and 2 of Definition 23, so $B_a \subseteq B_b \sqcup N_b$. Since $B_a \cap B_b = \emptyset$ by assumption, we must have $B_a \subseteq N_b$. But, according to Part 1 of Definition 23, N_b is completely joined to B_b, so B_a must be completely joined to B_b, as was to be shown.

It remains to be seen that $\mathscr{B}$ is a subpartition of $M_x \sqcup B_x \sqcup N_x$ with at least two elements. By hypothesis, B_x is non-empty, so there is at least one element in $\mathscr{B}$. It is also assumed that N_x is non-empty, so let $y \in N_x$. Then B_y is distinct from B_x and $\mathscr{B}$ has at least two elements. Since y, as an element of N_x, is completely joined to B_x, the previous paragraph indicates that B_y must be completely joined to B_x. This means that B_y must be contained in N_x, hence $y \in N_x$ implies $B_y \subseteq N_x$. Similarly, if $y \in M_x$, then B_y must be isolated from B_x, forcing $B_y \subseteq M_x$, so $\mathscr{B}$ is indeed a subpartition of $M_x \sqcup B_x \sqcup N_x$. □

Lemma 26. *Let Γ be a vertex transitive G-graph with a wreath product inducer as in Lemma 24, and let $\mathscr{B}$ be as in Lemma 24. Then for any $x \in V\Gamma$, $\Gamma \cong \Gamma_{\mathscr{B}} \wr \mathrm{Gr}(B_x)$, where $\Gamma_{\mathscr{B}}$ is a vertex transitive G-graph, $\mathrm{Gr}(B_x)$ is a vertex transitive H-graph for some $H \leqslant G_{N_x}$, and the factorization is non-trivial if and only if B_x contains more than one vertex.*

Proof. According to Lemmas 24 and 25, $V\Gamma$ is partitioned into a block system $\mathscr{B}$ for the G-action on $V\Gamma$ in such a way that any two blocks are either isolated from each other or are completely joined. Since some element of G carries any given block to any other given block, the subgraphs induced by all blocks are isomorphic and the graph $\Gamma_{\mathscr{B}}$ is a vertex transitive G-graph. $\mathrm{Gr}(B_x)$ is vertex transitive for some subgroup of $G_{N(x)}$ by Part 4 of Definition 23. According to Proposition 1, we have the claimed decomposition $\Gamma \cong \Gamma_{\mathscr{B}} \wr \mathrm{Gr}(B_x)$. Since $\mathscr{B}$ has at least two elements, $\Gamma_{\mathscr{B}}$ must have at least two vertices. It follows that $\Gamma_{\mathscr{B}} \wr \mathrm{Gr}(B_x)$ is non-trivial if and only if B_x contains more than one vertex of Γ. □

Lemma 27. *Let Γ be a vertex transitive G-graph with a wreath product inducer as in Lemma 24, and let $\mathscr{B}$ be as in Lemma 24. If, for some $x_0 \in V\Gamma$, $M_{x_0} = \emptyset$, then for all $x \in V\Gamma$, $M_x = \emptyset$ and $\Gamma_{\mathscr{B}} \cong K_m$ for some integer $m \geqslant 2$.*

Proof. We first argue that, regardless of whether or not M_x is empty, $gM_x = M_{gx}$ and $gN_x = N_{gx}$. To carry out this argument, consider two partitions of $V\Gamma$. The first partition is $V\Gamma = gM_x \sqcup gN_x \sqcup gB_x$. Since g is a graph automorphism, gB_x must be completely joined to gN_x and must be isolated from gM_x. The second partition is $V\Gamma = M_{gx} \sqcup N_{gx} \sqcup B_{gx}$, in which, by definition, B_{gx} is completely joined to N_{gx} and is isolated from M_{gx}. Since $gB_x = B_{gx}$ by Part 3 of Definition 23, $gM_x \sqcup gN_x = M_{gx} \sqcup N_{gx}$ and the join information forces $gM_x = M_{gx}$ and $gN_x = N_{gx}$ as claimed.

Now if $M_{x_0}=\emptyset$ for some $x_0 \in V\Gamma$, the results above show that $M_{gx_0}=\emptyset$ for all $g\in G$. Since Γ is vertex transitive, it follows that $M_x=\emptyset$ for all $x\in V\Gamma$. Knowing this, suppose B_y and B_z are two distinct blocks in $\mathscr{B}$. Since $\mathscr{B}$ is a subpartition of $B_y \sqcup N_y$, we must have $B_z \subseteq N_y$. N_y is completely joined to B_y, so B_z must be too. We already know that $\mathscr{B}$ consists of at least two blocks, and now we have established that each pair of blocks is completely joined. This means that $\Gamma_{\mathscr{B}} \cong K_m$ for some $m \geqslant 2$. □

10. Wreath product structure theorems

Although our primary interest is in connected graphs, a disconnected vertex transitive graph can be a wreath factor of a connected vertex transitive graph. Disconnected vertex transitive graphs have been characterized as wreath products in Lemma 16. As a corollary to that result, we add the following description of disconnected neighbor disconnected vertex transitive graphs.

Theorem 28. *Γ is a disconnected vertex transitive graph with disconnected survival subgraph if and only if $\Gamma \cong E_t \wr \Delta$ and either Δ is not a clique or, if Δ is a clique, then $t \geqslant 3$.*

Proof. Let Γ be a disconnected vertex transitive graph with disconnected survival subgraph. In view of Lemmas 16 and 8,

$$S\Gamma \cong SE_t \wr \Delta \sqcup S\Delta \cong E_{t-1} \wr \Delta \sqcup S\Delta,$$

where $t\geqslant 2$. By assumption, $S\Gamma$ must consist of at least two components. Thus, if either summand of the above disjoint union is empty, then the other summand must be disconnected. Since we are assuming $t\geqslant 2$, the only way a summand can be empty is if Δ is a clique. In this case, we must have $t\geqslant 3$ in order to ensure that the survival subgraph $E_{t-1} \wr \Delta$ is disconnected. □

We can now consider vertex transitive neighbor disconnected graphs in general.

Theorem 29. *Γ is a vertex transitive G-graph having $S\Gamma$ disconnected with one maximum component if and only if for some $t\geqslant 1, \Gamma \cong \Sigma \wr E_t \wr \Delta$, where Σ, $S\Sigma$, and Δ are connected, Σ is a vertex transitive H_1-graph, Δ is a vertex transitive H_2-graph, and H_1 and H_2 are subgroups of G.*

Proof. First, let Γ be a connected vertex transitive G-graph having $S\Gamma$ disconnected with one maximum component. Since $S_x\Gamma$ is disconnected, it must have at least one submaximum component. Since the vertices of all submaximum components lie in $I(U_x)$, $I(U_x)\neq\{x\}$. We claim that Γ has the wreath product inducer $M_x=U_x$, $B_x=I(U_x)$, $N_x=N(U_x)$, so we have to check that Definition 23 is satisfied. Since $S_x\Gamma$ is disconnected, $U_x\neq\emptyset$. Since Γ is connected, $N(U_x)\neq\emptyset$. We have a partition $V\Gamma=U_x\sqcup I(U_x)\sqcup N(U_x)$ for all $x\in V\Gamma$ by Lemma 5 Part 3. Thus the requirement that there is a partition $V\Gamma = M_x \sqcup B_x \sqcup N_x$, with B_x and N_x non-empty, is satisfied, and we turn to the verification of Parts 1–4 of Definition 23.

Part 1: $I(U_x)$ is isolated from U_x by definition, and Lemma 20 shows $I(U_x)$ is completely joined to $N(U_x)$.

Part 2: $x \in I(U_x)$ for all $x \in V\Gamma$ by definition of $I(U_x)$.

Part 3: $I(U_{gx}) = gI(U_x)$ by Lemma 15.

Parta 4: $G_{I(U_x)}$ is a subgroup $G_{N(U_x)}$ that operates transitively on $I(U_x)$ by Lemma 19.

Lemma 26 now gives us a wreath product decomposition $\Gamma \cong \Sigma \wr \Phi$ with Σ a vertex transitive G-graph, Φ a vertex transitive L-graph for a subgroup $L \leqslant G$, and $\Phi \cong \mathrm{Gr}(B_x) \cong \mathrm{Gr}(I(U_x))$. This factorization is non-trivial since $I(U_x) \neq \{x\}$. If Σ is a clique, then $S\Sigma = \emptyset$ and so is connected. If Σ is not a clique, the formula of Lemma 8 indicates that the maximum components of $\Sigma \wr \Phi$ must be contained in $S\Sigma \wr \Phi$. Since there is only one maximum component, and since the vertices of all submaximum components are in $I(U_x)$, $S\Sigma \wr \Phi$ must be the maximum component and all submaximum components are components of $S\Phi$. Consequently, $S\Sigma$ must be connected. Φ is a vertex transitive L-graph which may or may not be connected, so write $\Phi \cong E_t \wr \Delta$ where Δ is a connected vertex transitive H_2-graph by Lemma 16. We can take $H_1 = G$ and $H_2 \leqslant L \leqslant G$, and this completes the proof of necessity for connected Γ.

Second, if Γ is disconnected, we can, according to Lemma 16, write $\Gamma \cong K_1 \wr E_t \wr \Delta$ and this obviously satisfies the conclusion of the theorem.

Conversely, suppose that for some $t \geqslant 1$, $\Gamma \cong \Sigma \wr E_t \wr \Delta$ where Σ, $S\Sigma$, and Δ are connected, Σ is a vertex transitive H_1-graph and Δ is a vertex transitive H_2-graph. Then, letting S_t denote the symmetric group on t elements, Γ is a connected vertex transitive $H_1 \wr S_t \wr H_2$-graph and it is easy to see, using Proposition 10 and the fact that $S\Sigma$ is connected, that $S\Gamma$ has a single maximum component. □

Remark. In the decomposition $\Gamma \cong \Sigma \wr E_t \wr \Delta$ obtained above, it is possible that Δ consists of just a single vertex. Since $S\Gamma$ is disconnected and $S\Sigma$ is connected, we must have $t \geqslant 2$ in this case. $S\Gamma$ will have a single maximum component isomorphic to $S\Sigma \wr E_t$ and $t-1$ isolated vertices. If Δ is a single vertex, then Σ cannot be a clique, otherwise we would have $S\Gamma \cong E_{t-1}$ with $t \geqslant 3$, and so the survival graph would have more than one maximum component. The situation when Δ is a single vertex and Σ is a clique is handled in Theorem 17 as well as the next theorem.

Theorem 30. *Γ is a vertex transitive G-graph having $S\Gamma$ disconnected with more than one maximum component if and only if for some $m \geqslant 1$ and $t \geqslant 3$, $\Gamma \cong K_m \wr E_t \wr \Delta$, where Δ is a connected vertex transitive H-graph for some $H \leqslant G$. In this case, $S\Gamma$ has $t-1$ maximum components, each isomorphic to Δ, and if $S\Gamma$ has no submaximum components, then Δ is a clique.*

Proof. Assume first that Γ is a connected vertex transitive G-graph having $S\Gamma$ disconnected with more than one maximum component. We could repeat the argument for

Theorem 29 to get a wreath product decomposition $\Sigma \wr \Delta$ with $\Delta \cong \mathrm{Gr}(I(U_x))$. However, Γ no longer has to have submaximum components, so it is possible that $I(U_x) = \{x\}$, in which case we would end up with the trivial wreath product decomposition. The cure is to find a B_x that properly contains $I(U_x)$, so that it does not matter whether or not $I(U_x)$ has only one element. It is possible to find such a B_x whenever $S\Gamma$ has more than one maximum component, as we now show.

We obtain a wreath product inducer by taking $M_x = \emptyset$, $B_x = U_x \sqcup I(U_x)$, $N_x = N(U_x)$, and then checking that the conditions of Definition 23 are satisfied. U_x and $N(U_x)$ are non-empty for the same reason as they were in Theorem 29, so B_x and N_x are non-empty. We have a partition $V\Gamma = M_x \sqcup B_x \sqcup N_x$ for all $x \in V\Gamma$, because this is just a differently associated version of the disjoint union used in the proof of Theorem 29. We can turn to the verification of Parts 1–4 of Definition 23.

Part 1: The empty set of vertices is isolated from $U_x \sqcup I(U_x)$, and Lemmas 20 and 21 show that $U_x \sqcup I(U_x)$ is completely joined to $N(U_x)$.

Part 2: $x \in I(U_x)$ for all $x \in V\Gamma$ by definition of $I(U_x)$, so certainly $x \in U_x \sqcup I(U_x)$ for all $x \in \Gamma$.

Part 3: $U_{gx} \sqcup I(U_{gx}) = g(U_x \sqcup I(U_x))$ by Lemma 15.

Part 4: This is Conclusion 3 of Lemma 22.

Lemmas 26 and 27 now give us a wreath product decomposition $\Gamma \cong K_m \wr \Phi$, with $m \geqslant 2, \Phi$ a vertex transitive H-graph for some $H \leqslant G$, and $\Phi \cong \mathrm{Gr}(B_x) \cong \mathrm{Gr}(U_x \sqcup I(U_x))$. B_x has at least two elements, because U_x is non-empty and $I(U_x)$ has x in it at the very least, so the factorization is non-trivial.

Φ is a vertex transitive graph that may or may not be connected, so, using Lemma 16, we write $\Phi \cong E_t \wr \Delta$, where Δ is a connected vertex transitive graph. We then have

$$\begin{aligned} S\Gamma &\cong SK_m \wr E_t \wr \Delta \sqcup SE_t \wr \Delta \sqcup S\Delta, \\ &\cong E_{t-1} \wr \Delta \sqcup S\Delta, \end{aligned}$$

so $S\Gamma$ has $t-1 \geqslant 2$ maximum components, each isomorphic to Δ. Furthermore, if $S\Gamma$ has no submaximum components, then we know from Lemma 22 that Δ must be a clique. This completes the proof of necessity for connected Γ.

If Γ is disconnected but satisfies the other hypotheses, the wreath product inducer arguments do not apply because $N_x = \emptyset$. But by Lemma 16, we have $\Gamma \cong K_1 \wr E_t \wr \Delta$, hence $S\Gamma \cong E_{t-1} \wr \Delta \sqcup S\Delta$ just as above, and the rest of the argument proceeds identically.

Conversely, suppose that for some $m \geqslant 1$ and $t \geqslant 3$, $\Gamma \cong K_m \wr E_t \wr \Delta$, where Δ is a connected vertex transitive H-graph. As we have argued above, Γ has more than one maximum component. Furthermore, Γ is a vertex transitive $S_{mt} \wr H$-graph by Theorems 29 and 30, Lemma 16, and the fact that $S_m \wr S_t = S_{mt}$. $\square$

Remark. In the decomposition $\Gamma \cong K_m \wr E_t \wr \Delta$ obtained above, it is possible that Δ consists of just a single vertex, in which case $S\Gamma = E_{t-1}$ with $t \geqslant 3$. It is even possible that $m = 1$ in which case $\Gamma = E_t$. These special cases were originally handled by

Theorem 17. What is important to note now is that empty graphs with at least three vertices are the only vertex transitive graphs that are neighbor disconnected but do not have a non-trivial wreath product decomposition.

In the next theorem, we combine Theorems 29, 30 and Lemma 16 into one inclusive, if less detailed, result asserting that Γ is a wreath product of a certain type if and only if Γ is neighbor disconnected.

Theorem 31. *Γ is a non-empty neighbor disconnected vertex transitive G-graph if and only if Γ has a non-trivial wreath product factorization $\Gamma \cong \Lambda \wr \Omega$ in which both Λ and Ω are vertex transitive for some subgroups of G, and if one of Λ and Ω is neighbor connected, then the other cannot be a clique.*

Proof. Depending on the number of maximum components of $S\Gamma$, either Theorem 29 or Theorem 30 applies and gives a wreath product factorization, which is non-trivial since Γ is non-empty. The final condition imposed on Λ and Ω is necessary and sufficient by Corollary 9. □

The next result describes the behavior of the above wreath product factorizations with respect to their full automorphism groups. In general, given a wreath product factorization $\Gamma \cong \Lambda \wr \Omega$, it is always true that $\mathrm{Aut}(\Lambda) \wr \mathrm{Aut}(\Omega) \leqslant \mathrm{Aut}(\Gamma)$, but there are further conditions that must be met before equality is assured (see [6]). In general, Γ may have automorphisms that do not respect the partition of $V\Gamma$ that corresponds to a given wreath product factorization. The following theorem indicates that such 'extra' automorphisms are not present in the wreath product factorizations obtained above.

Theorem 32. *Let Γ be a non-empty neighbor disconnected vertex transitive graph and $\Gamma \cong \Lambda \wr \Omega$ the non-trivial factorization provided by Theorem* 31. *Then* $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Lambda) \wr \mathrm{Aut}(\Omega)$.

Proof. We have to remember where Λ and Ω come from. If Γ is disconnected, $\Lambda \cong E_t$, Ω is isomorphic to any of the components of Γ, and the result is elementary in this case. If Γ is connected, then we showed in Theorems 29 and 30 that $\Gamma \cong \Gamma_{\mathscr{B}} \wr \mathrm{Gr}(B_x)$, where $\mathscr{B}$ is a non-trivial partition of $V\Gamma$ that depends on the number of maximum components in $S\Gamma$. Here $\Lambda \cong \Gamma_{\mathscr{B}}$ and $\Omega \cong \mathrm{Gr}(B_x)$. The partitioning sets in $\mathscr{B}$ are the sets $B_x = I(U_x)$ when there is only one maximum component, and $B_x = U_x \sqcup I(U_x)$ when there is more than one maximum component. In either case, the sets B_x are an $\mathrm{Aut}(\Gamma)$-equivariant partition of $V\Gamma$ by Lemma 15, so $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Gamma_{\mathscr{B}}) \wr \mathrm{Aut}(B_x)$ by Proposition 1. This proves the theorem. □

Remark. Care must be exercised when applying Theorem 32. In particular, the statement that $\Lambda \wr \Omega$ must be 'the factorization provided by Theorem 31' cannot be ignored. More specifically, Ω must be $I(U_x)$ or $U_x \sqcup I(U_x)$, and must be the second possibility if $I(U_x) = \{x\}$.

For example, consider $\Gamma = C_4 \wr E_2$. We have $\mathrm{Aut}(C_4) \wr \mathrm{Aut}(E_2) \cong D_4 \wr \mathbb{Z}_2$. This group, whose order is $8 \cdot 2^4 = 128$, is not even close to the full automorphism group of Γ, but Theorem 32 is not contradicted because the factorization $C_4 \wr E_2$ is not the one that is provided by Theorem 31. To see this, use the process of Theorem 31 to compute a wreath product factorization of Γ. For any $x \in V\Gamma$, $S_x\Gamma \cong SC_4 \wr E_2 \sqcup SE_2 \cong E_3 \cong U_x$, and $I(U_x) = \{x\}$, so we must invoke Theorem 30 and set $B_x = U_x \sqcup I(U_x) \cong E_4$. We then see that $\Gamma_{\mathscr{B}} = K_2$. Thus, the wreath product factorization of Γ provided by Theorem 31 is $\Gamma = K_2 \wr E_4$, and for this factorization we know that $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(K_2) \wr \mathrm{Aut}(E_4) \cong \mathbb{Z}_2 \wr S_4$, a group of order $2 \cdot 24^2 = 1152$. This group is much bigger than the subgroup of order 128 we found from the original wreath product factorization of Γ.

What is going on here is that $\Gamma \cong K_2 \wr E_2 \wr E_2$, we defined Γ to be $(K_2 \wr E_2) \wr E_2$, the factorization provided by Theorem 31 is $K_2 \wr (E_2 \wr E_2)$, and Aut does not respect the associative law for wreath products unless the conditions of [6] are met. It is interesting that the process of Theorem 31 provides a method to detect and reassociate wreath factorizations that are not adapted to the computation of $\mathrm{Aut}(\Gamma)$.

For our final general structure theorem, we once again combine Theorems 29, 30, and Lemma 16, this time using them repeatedly to produce a factorization into 'indecomposables', in which the factors are either neighbor connected or empty.

Theorem 33. *Let Γ be a vertex transitive G-graph. Then Γ is neighbor disconnected if and only if Γ has a wreath product factorization in which the factors are either empty graphs or connected neighbor connected graphs. The factorization has the form*

$$\Gamma \cong \Sigma_1 \wr E_{t_1} \wr \Sigma_2 \wr E_{t_2} \wr \cdots \wr \Sigma_{k-1} \wr E_{t_{k-1}} \wr \Sigma_k$$

where each Σ_i is a connected neighbor connected vertex transitive H_i-graph for some $H_i \leqslant G, |\Sigma_i| \geqslant 2$ for $2 \leqslant i \leqslant k-1$, and $|\Sigma_i| \geqslant 1$ for $i = 1, k$.

Proof. Sufficiency is, by now, obvious and we concern ourselves first with the necessity of the factorization formula. Thus we begin with Γ a neighbor disconnected vertex transitive G-graph. According to Theorems 29 and 30, $\Gamma \cong \Sigma_1 \wr E_{t_1} \wr \Delta_1$, where $t_1 \geqslant 1$, Σ_1 is a connected neighbor connected vertex transitive graph for some subgroup of G, and Δ_1 is a connected vertex transitive graph of order at least one for some subgroup of G.

If Δ_1 is neighbor connected, then no further factorization will take place and we set $\Sigma_2 = \Delta_1$ and so have the result of the theorem with $k = 2$. If Δ_1 is not neighbor connected, then we use Theorems 29 and 30 on Δ_1, obtaining $\Delta_1 \cong \Sigma_2 \wr E_{t_2} \wr \Delta_2$, with Σ_2, Δ_2 and t_2 satisfying the same conditions as Σ_1, Δ_1, and t_1 respectively, except that Σ_2 must be of order at least two because Δ_1 is connected. If Δ_2 is neighbor connected, then no further factorization will take place and we set $\Sigma_3 = \Delta_2$ and so have the result of the theorem with $k = 3$. Otherwise, we continue until finally we reach Δ_{k-1} neighbor connected as well as connected and vertex transitive. The process must end this way because the orders of the graphs Δ_i are a strictly decreasing sequence. We then set $\Sigma_k = \Delta_{k-1}$ and obtain the formula of the theorem. $\square$

11. Neighbor connected graphs with clique survival subgraphs

We comment here on the fact that our wreath product characterization of neighbor disconnected graphs cannot be extended to graphs of neighbor connectivity one. The vertex transitive graphs of neighbor connectivity one that are neighbor connected are either cliques or are connected with survival subgraphs that are cliques, and the latter class contains infinitely many graphs with no wreath product decomposition. For example, take $n \geqslant 2$ and $\Gamma = \mathrm{Cay}(\mathbb{Z}_{3n-1}, \{\pm 1, \pm 2, \ldots, \pm(n-1)\})$. We have $VS_0\Gamma = \{n,\ n+1,\ n+2, \ldots, 2n-1\}$, and the generators $\pm 1, \ldots, \pm(n-1)$ connect every pair of these surviving vertices, thereby creating a clique of order n. There are infinitely many values of n for which $3n-1$ is prime, and for all such values, Γ cannot be a non-trivial wreath product.

Thus, although $\Gamma \cong K_m \wr E_2 \wr K_n$ with $m, n \geqslant 1$ is sufficient for $S\Gamma \cong K_n$, and necessary and sufficient when $n = 1$ by Theorem 17, as soon as $n \geqslant 2$, we can no longer assert that a vertex transitive Γ with $S\Gamma = K_n$ must have any kind of non-trivial wreath product factorization.

A way to gauge the profusion of different vertex transitive graphs Γ with a clique survival subgraphs is to use Lemma 5, Part 4 to observe that it is equivalent to require that $\mathscr{C}\Gamma$ is a triangle-free vertex transitive graph, and there is an extremely large and varied supply of these.

12. Applications

We restrict ourselves to the most elementary applications of the theorems of the previous section.

Theorem 34. *Let Γ be a non-empty vertex transitive G-graph with primitive G-action. Then Γ is connected and neighbor connected.*

Proof. The components of a non-empty disconnected vertex transitive G-graph are automatically blocks for the G-action, which must, therefore, be imprimitive. Hence, Γ is connected. The wreath product decompositions in Theorems 29 and 30 both correspond to non-trivial G-equivariant partitions of $V\Gamma$, as long as we are not in the degenerate case of Theorem 30 when $\Gamma \cong E_t$. Thus, G is imprimitive whenever the survival subgraph of Γ is disconnected, so Γ must be neighbor connected. □

Corollary 35. *Non-empty prime order circulants are connected and neighbor connected.*

Proof. A prime order circulant is a vertex transitive graph with a regular action of $\mathbb{Z}_p$ for some prime p, and all regular $\mathbb{Z}_p$ actions are primitive. □

This corollary appears in [DGS] as part of Corollary 15.1. It can also be obtained as a corollary to Proposition 37 below.

Proposition 36. *Γ is a connected, neighbor disconnected vertex transitive graph whose order is the product of two primes if and only if Γ is a wreath product having one of the following three forms: Let Λ and Ω denote prime order circulants that are neither empty graphs nor cliques. Then either*

1. *$\Gamma \cong \Lambda \wr \Omega$, or*
2. *$\Gamma \cong \Lambda \wr E_q$ with q prime, or*
3. *$\Gamma \cong K_p \wr E_q$ with p prime and q an odd prime.*

Proof. It is easily verified that the three given conditions are sufficient for Γ to be a connected, neighbor disconnected vertex transitive graph, so we proceed to the proof of necessity, in which almost every statement is justified by Theorem 31.

We know that Γ must be a wreath product of prime order circulants Λ and Ω. Since Γ is connected, Λ cannot be an empty graph and so must be neighbor connected by Corollary 35. Consequently, Ω cannot be a clique. If Ω is not an empty graph, than it too is neighbor connected and so in this case Λ cannot be a clique. If Ω is an empty graph E_q, then Λ can be a clique, but if it is, then $S\Gamma \cong E_{q-1}$, which shows that q cannot be 2 in this case. □

A number of general results on neighbor connectedness are a consequence of the formula $\deg(\Sigma \wr \Delta) = \deg(\Sigma)|\Delta| + \deg(\Delta)$. We give two examples below.

Proposition 37. *Let Γ be a connected vertex transitive graph and p the smallest prime dividing the order of Γ.*

1. *If $\deg(\Gamma) < p$, then Γ is neighbor connected.*
2. *If $\deg(\Gamma) = p$, then either Γ is neighbor connected or else $\Gamma \cong K_{p,p}$.*
3. *If $\deg(\Gamma) = p+1$, then either Γ is neighbor connected or else $\Gamma \cong K_{p+1,p+1}$.*

Proof. If Γ is neighbor disconnected, then Γ has a non-trivial wreath product decomposition $\Gamma \cong \Sigma \wr \Delta$ with Σ connected. Since Σ is connected and cannot be K_1, $\deg(\Sigma) \geqslant 1$. We have $\deg(\Sigma)|\Delta| + \deg(\Delta) = \deg(\Gamma)$. Since $|\Delta|$ is a divisor of $|\Gamma|$, we must have $|\Delta| \geqslant p$. We now consider the three cases in the proposition.

Case 1: The degree formula cannot hold with $|\Delta| \geqslant p$ and $\deg(\Gamma) < p$, so Γ must be neighbor connected in this case.

Case 2: The only way the degree formula can hold with $|\Delta| \geqslant p$ and $\deg(\Gamma) = p$ is if $|\Delta| = p$, $\deg(\Sigma) = 1$, $\deg(\Delta) = 0$. This requires Σ to be a connected vertex transitive graph of degree 1, so $\Sigma \cong K_2$. Δ must be a vertex transitive graph of order p and degree 0, so $\Delta \cong E_p$. Thus $\Gamma \cong K_2 \wr E_p = K_{p,p}$.

Case 3: There are two ways the degree formula can hold with $|\Delta| \geqslant p$ and $\deg(\Gamma) = p+1$: either $|\Delta| = p$ or $|\Delta| = p+1$. Suppose $|\Delta| = p$. Then we must have $\deg(\Sigma) = 1$ and $\deg(\Delta) = 1$. The only vertex transitive graph Δ of order p and degree 1 is K_2 — i.e. p must be 2 in this case — and we also have $\Sigma = K_2$, so then $\Gamma \cong K_4$ which is, however, neighbor connected. Thus $|\Delta| = p$ cannot occur in this case if Γ is neighbor disconnected. This leaves us with $|\Delta| = p+1$, in which case we must have $\deg(\Sigma) = 1$ and $\deg(\Delta) = 0$, so $\Gamma \cong K_2 \wr E_{p+1} \cong K_{p+1,p+1}$. □

Table 1
All connected neighbor disconnected vertex transitive graphs of degree not exceeding 10

Deg	Graphs			
2	none	—	—	—
3	$K_{3,3}$	—	—	—
4	$K_{4,4}$	—	—	$C_m \wr E_2$
5	$K_{5,5}$	—	—	$C_m \wr K_2$
6	$K_{6,6}$	$\Sigma_3 \wr E_2$	$C_4 + C_4$	$C_m \wr E_3$
7	$K_{7,7}$	$\Sigma_3 \wr K_2$	$C_5 + C_5$	—
8	$K_{8,8}$	$\Sigma_4 \wr E_2$	$C_6 + C_6$	$C_m \wr E_4$
—	—	—	—	$C_m \wr K_3$
9	$K_{9,9}$	$\Sigma_3 \wr E_3$	$C_7 + C_7$	$C_m \wr E_2 \wr K_2$
—	$K_{4\times 3}$	$\Sigma_4 \wr K_2$	$(K_3 \times K_3) + (K_3 \times K_3)$	—
10	$K_{10,10}$	$\Sigma_5 \wr E_2$	$C_8 + C_8$	$C_m \wr C_4$
—	$K_{6\times 2}$	—	—	$C_m \wr E_5$

Proposition 37 implies that if $\Gamma = \mathrm{Cay}(G,S)$ for a subset $S \subseteq G$ that generates G, and if $|S|$ is less than the smallest prime dividing $|\Gamma|$, then Γ is neighbor connected. This result for Cayley graphs is obtained in Corollary 15.1 of [DGS], but with the additional assumptions that $G \backslash S$ generates G and that S is a union of conjugacy classes. Proposition 37 also implies that if $|S| = p$ and $\Gamma \ncong K_{p,p}$, then Γ is neighbor connected. A corresponding result for Cayley graphs is obtained in Corollary 15.2 of [DGS], but with the additional assumptions that G is abelian and $G \backslash S$ generates G. (The assumption that $G \backslash S$ must generate G is a stronger version of our exclusion of $K_{p,p}$.)

For our second example of the applications of the degree formula, we continue in the spirit of the proof of Proposition 37 and compile a small table, Table 1 above, of all connected neighbor disconnected vertex transitive graphs of small degree. In the table, we use the notation Σ_d to denote an arbitrary connected vertex transitive graph of degree d. We write the complete join $\Gamma + \Gamma$ for the wreath product $K_2 \wr \Gamma$. The cartesian product of graphs is symbolized by '$\times$', as is customary. For complete multipartite graphs $K_{t,t,\dots,t}$ in which the 't' is repeated n times, we write $K_{n\times t}$, except that we continue to write $K_{t,t}$ rather than $K_{2\times t}$. For wreath products of the form $C_m \wr \Delta$, it is assumed that $m \geqslant 3$, unless Δ is a clique, in which case it is assumed that $m \geqslant 4$.

We illustrate how the entries corresponding to deg $= 9$ of this table were obtained. The other rows are found in a similar manner. Let Γ be a connected neighbor disconnected vertex transitive graph. Then Γ has a non-trivial wreath product decomposition $\Gamma \cong \Sigma \wr \Delta$ in which Σ and Δ are vertex transitive graphs and Σ is connected. Let $\delta_\Sigma = \deg(\Sigma)$, $D = |\Delta|$, and $\delta_\Delta = \deg(\Delta)$. These three natural numbers must satisfy the following conditions:

1. $\delta_\Sigma \geqslant 1$, since Σ is connected and is a non-trivial wreath factor of Γ.
2. $D \geqslant 2$, since Δ is a non-trivial wreath factor of Γ.
3. $0 \leqslant \delta_\Delta < D$.
4. $D\delta_\Delta$ must be even, because it is $2|E\Delta|$.

In addition to these conditions, we use the following pair of facts: The only vertex transitive graphs of degree 1 are disjoint unions of K_2's, i.e. graphs of the form $E_t \wr K_2$, and the only vertex transitive graphs of degree 2 are disjoint unions of cycle graphs, i.e. graphs of the form $E_t \wr C_m$.

Now suppose $\deg(\Gamma)=9$. We have to consider all possible numerical values of the parameters δ_Σ, D, and δ_Δ that satisfy the above four conditions and the condition $\delta_\Sigma D + \delta_\Delta = 9$. A systematic way to do this is to first partition 9 into summands $P + \delta_\Delta = 9$ with $P > \delta_\Delta$ (since D is a factor of P and $D > \delta_\Delta$), then factor P into all products $\delta_\Sigma \cdot D = P$, and then consider the graphs realizing the allowable combinations of δ_Σ, D, and δ_Δ. The partitions of 9 that must be considered are $9+0$, $8+1$, $7+2$, $6+3$, and $5+4$. Of the fifteen possibilities that occur when 9, 8, 7, 6, and 5 are factored to provide possible values for δ_Σ and D, $9 \cdot 1$, $8 \cdot 1$, $7 \cdot 1$, $6 \cdot 1$, and $5 \cdot 1$ are eliminated by Condition 2. Two more possibilities, $3 \cdot 2+3$ and $2 \cdot 3+3$ are eliminated by Condition 3. (Condition 4 is not needed for the case $\deg(\Gamma)=9$, although it is used to eliminate possible factorizations for other degrees.) All other cases yield possible graphs for the table. We consider them in turn:

$3 \cdot 3+0$: This gives $\Sigma_3 \wr E_3$.

$1 \cdot 9+0$: This gives $K_2 \wr E_9 \cong K_{9,9}$.

$4 \cdot 2+1$: This gives $\Sigma_4 \wr K_2$.

$2 \cdot 4+1$: This gives $C_m \wr (E_2 \wr K_2)$ for $m \geqslant 3$.

$1 \cdot 8+1$: This gives $K_2 \wr (E_4 \wr K_2) \cong K_{4,4} \wr K_2$, which can be subsumed under $\Sigma_4 \wr K_2$.

$1 \cdot 7+2$: This gives $K_2 \wr C_7$.

$1 \cdot 6+3$: This gives $K_2 \wr \Delta$, where Δ is a vertex transitive graph of order 6 and degree 3. $\mathscr{C}\Delta$ must then be a vertex transitive graph of order 6 and degree 2. Thus, either $\mathscr{C}\Delta \cong C_6$ or $\mathscr{C}\Delta \cong E_2 \wr K_3$, which means that either $\Delta \cong K_3 \times K_3$ or $\Delta \cong K_2 \wr E_3$ respectively. The possibilities for this case are thus $K_2 \wr (K_3 \times K_3) \cong (K_3 \times K_3) + (K_3 \times K_3)$ and $K_2 \wr K_2 \wr E_3 \cong K_4 \wr E_3 = K_{4\times 3}$.

$1 \cdot 5+4$: This gives $K_2 \wr K_5 \cong K_{10}$, which is neighbor connected and so does not appear in the table.

13. Neighborhood structure theorems

The theorems that follow refine and generalize the results in [DGS] on the necessity and sufficiency of periodic and nearly periodic generating sets for disconnection of the survival subgraph of a Cayley graph. The results of this section apply to all vertex transitive graphs, whereas the corresponding results in [DGS] are proved only for a restricted class of Cayley graphs.

It is useful, as in [DGS], to analyze disconnection not only in terms of the 'biggest' components of the survival subgraph, as we have done in Theorems 29 and 30, but also in terms of the 'smallest' components. The next proposition describes the number of isolated vertices that can appear in the survival subgraph of a vertex transitive graph.

Proposition 38. *Let Γ be a vertex transitive G-graph. Then the number of isolated vertices of $S_x\Gamma$ is $[G_{N(x)} : G_x] - 1$.*

Proof. An arbitrary vertex of Γ has the form gx for some $g\in G$, and

$$\begin{aligned} gx=x \text{ or is isolated in } S\Gamma &\Leftrightarrow N(gx)=N(x),\\ &\Leftrightarrow gN(x)=N(x), \quad \text{by Proposition 12,}\\ &\Leftrightarrow g\in G_{N(x)}. \end{aligned}$$

Thus, $x\sqcup\{$the isolated vertices of $S\Gamma\}$ is the orbit of x under $G_{N(x)}$, and the order of this orbit is $[G_{N(x)}:G_x]$. Deleting x from the orbit, we obtain the lemma. $\square$

According to this proposition, $G_{N(x)}$ properly containing G_x is necessary and sufficient for the presence of at least one isolated vertex in the survival subgraph.

We now begin a sequence of lemmas that combine to yield the main theorem of the section, Theorem 43, which gives necessary and sufficient conditions for neighbor disconnection in terms of neighborhoods and orbits of the center of deletion. The basic idea behind the theorem is that if Γ is a vertex transitive G-graph, then it is always possible to find a subset $L\subseteq N(x)$ and a subgroup $H\leqslant G_L$ such that $N[x]\subseteq L\sqcup Hx\subseteq V\Gamma$, for example $L=N(U_x)$ and $H=G_{N(U_x)}$ will do. Roughly speaking, Γ is neighbor disconnected if and only if it is possible to choose L and H so that one or both of the above inclusions is proper. (Consult Theorem 43 for the qualifications needed when only one of the inclusions is proper.)

The first lemma in the sequence gives a pair of necessary conditions for neighbor disconnection of a vertex transitive G-graph, and the second lemma indicates that a single condition is all that is needed if $G=\mathrm{Aut}(\Gamma)$.

Lemma 39. *Let Γ be a neighbor disconnected vertex transitive G-graph. Then either*
1. $N[x]\subsetneq N(U_x)\sqcup G_{I(U_x)}x\subsetneq V\Gamma$, *or*
2. $N[x]=N(U_x)\sqcup G_{I(U_x)}x\subsetneq V\Gamma$ *and* $[G_{N(U_x)}:G_{I(U_x)}]\geqslant 3$.

If Condition 2 holds, then $\Gamma\cong K_m\wr E_t\wr K_n$, with $m\geqslant 1$, $t=[G_{N(U_x)}:G_{I(U_x)}]$, and $n=|G_{I(U_x)}x|$.

Proof. By Lemma 19, $G_{I(U_x)}\leqslant G_{N(U_x)}$ and $I(U_x)=G_{I(U_x)}x$. It is always the case that $N[x]\subseteq N(U_x)\sqcup I(U_x)\subseteq V\Gamma$. We now consider two cases:

Case 1: $I(U_x)$ is not a clique. Since $I(U_x)$ contains vertices not adjacent to x, we must have $N[x]\subsetneq N(U_x)\sqcup I(U_x)$. Since $U_x\neq\emptyset$ by the hypothesis of neighbor disconnection, we also have the proper inclusions $N(U_x)\sqcup I(U_x)\subsetneq N(U_x)\sqcup I(U_x)\sqcup U_x=V\Gamma$, and so we have Condition 1.

Case 2: $I(U_x)$ is a clique K_n, $n\geqslant 1$. It is immediate that $N[x]=N(U_x)\sqcup I(U_x)$ and that Γ can have no submaximal components, so Γ satisfies the hypotheses of Lemmas 21 and 22. These lemmas tell us that U_x is a union of cliques K_n, each of the form $I(U_{gx})$ for some $g\in G$, and that $G_{N(U_x)}$ acts transitively on $U_x\sqcup I(U_x)$. This action permutes the induced set of cliques since $gI(U_x)=I(U_{gx})$. The stabilizer for the action of $G_{N(U_x)}$ on this set of cliques is $G_{I(U_x)}$, and in order for $S\Gamma$ to be disconnected, the orbit must contain $\mathrm{Gr}(I(U_x))$ and at least two other cliques in $\mathrm{Gr}(U_x)$, hence $[G_{N(U_x)}:G_{I(U_x)}]\geqslant 3$. We thus have Condition 2.

According to Theorem 30, $\Gamma \cong K_m \wr E_t \wr K_n$ with $t-1$ the number of cliques in $S\Gamma$, making $t=[G_{N(U_x)} : G_{I(U_x)}]$. Since each clique is isomorphic to $\mathrm{Gr}(I(U_x))$, we have $n=|G_{I(U_x)}x|$, verifying the final claim of the lemma. $\square$

Lemma 40. *Let Γ be a neighbor disconnected vertex transitive G-graph with $G=\mathrm{Aut}(\Gamma)$. Then there is a subgroup $H \leqslant G_{N(U_x)}$ such that $N[x] \subsetneq N(U_x) \sqcup Hx \subsetneq V\Gamma$.*

Proof. If $I(U_x)$ is not a clique, the proof of Case 1 of Lemma 39 applies with $H=G_{I(U_x)}$. Suppose $I(U_x)$ is a clique K_n. We know from the proof of Lemma 39 that $\Gamma \cong K_m \wr E_t \wr K_n$ with $m,n \geqslant 1$ and $t \geqslant 3$.

Pick $y \in U_x$. Then, according to Lemma 22, $I(U_y)$ induces a clique in $\mathrm{Gr}(U_x)$ isomorphic to the clique induced by $I(U_x)$. Pick an isomorphism of $\mathrm{Gr}(I(U_x))$ with $\mathrm{Gr}(I(U_y))$, and let σ be the permutation of $V\Gamma$ that interchanges corresponding vertices of $I(U_x)$ and $I(U_y)$ and fixes all other vertices. Let P_x be the set of all permutations of the vertices of $V\Gamma$ that permute the vertices of $\mathrm{Gr}(I(U_x))$ and fix all other vertices, and let P_y be the set of all permutations of the vertices of $V\Gamma$ that permute the vertices of $\mathrm{Gr}(I(U_y))$ and fix all other vertices. Since $I(U_x)$ and $I(U_y)$ are completely joined to $N(U_x)$ and isolated from the remaining vertices in U_x, σ and the elements of P_x and P_y induce automorphisms of Γ. It is at this point that we need the hypothesis that $G=\mathrm{Aut}(\Gamma)$ to conclude that σ, the elements of P_x, and the elements of P_y all belong to G.

Let H be the subgroup of the permutation group of $V\Gamma$ generated by σ, P_x, and P_y. Then $H \leqslant G$ and moreover, since H actually fixes $N(U_x)$ pointwise, $H \leqslant G_{N(U_x)}$. We have $Hx=I(U_x) \sqcup I(U_y)$ and $N[x]=I(U_x) \sqcup N(U_x)$, so $N[x] \subsetneq N[x] \sqcup I(U_y)=N(U_x) \sqcup Hx$. Since $S_x\Gamma$ is disconnected, U_x must contain at least one more clique in addition to the one induced by $I(U_y)$, and so $N(U_x) \sqcup Hx \subsetneq V\Gamma$. $\square$

The next pair of lemmas establish general conditions (modeled on the necessary conditions in Lemma 39) that are sufficient for neighbor disconnection.

Lemma 41. *Let Γ be a vertex transitive G-graph. Then Γ is neighbor disconnected if there is a subset $L \subseteq N(x)$ and a subgroup $H \leqslant G_L$ such that $N[x] \subsetneq L \sqcup Hx \subsetneq V\Gamma$.*

Proof. The proper subset requirements guarantee that $(L \cup Hx) \backslash N[x]$ and $V\Gamma \backslash (L \cup Hx)$ are non-empty subsets of $VS_x\Gamma$. Moreover, these two sets are isolated from each other in $VS_x\Gamma$, thereby making $S_x\Gamma$ disconnected. To see this, it suffices to note that there can be no edge in Γ from a vertex $v \in (L \cup Hx) \backslash N[x]$ to a vertex $w \in V\Gamma \backslash (L \cup Hx)$. For suppose v and w are adjacent. From $L \subsetneq N[x]$ it follows that $(L \cup Hx) \backslash N[x] \subsetneq Hx$, so we can write $v=hx$ for some $h \in H$, and since hx is adjacent to w, we have x adjacent to $h^{-1}w$. $h^{-1}w$ is in $V\Gamma \backslash (L \cup Hx)$, because H, as a subgroup of G_L, stabilizes $L \cup Hx$ setwise. But $V\Gamma \backslash (L \cup Hx) \subsetneq V\Gamma \backslash N[x]=I(x)$, so we have x adjacent to something in $I(x)$, a contradiction that concludes the argument. $\square$

Remark. It is possible that $L=\emptyset$, in which case Γ must be disconnected. For if $L=\emptyset$, then Hx is an orbit that contains $N[x]$ but is not all of $V\Gamma$. Since Hx must contain the closed neighbood of all of its vertices, the vertices outside Hx must be isolated from Hx and so Γ is disconnected.

Lemma 42. *Let Γ be a vertex transitive G-graph. Then Γ is neighbor disconnected if there is a subset $L\subseteq N(x)$ and a subgroup $H\leqslant G_L$ such that $N[x]=L\sqcup Hx$ and $[G_L:(G_L)_{Hx}]\geqslant 3$.*

Proof. $\mathrm{Gr}(Hx)$ is a clique since $Hx\subseteq N[x]$. Moreover, Hx is completely joined to L, since $L\subseteq N(x)$ and H is transitive on Hx while stabilizing L. Since $N[x]=L\sqcup Hx$, it follows that $\deg(\Gamma)=|L|+|Hx|-1$.

Consider the translates gHx of Hx for $g\in G_L$: any such translate is the vertex set of a clique that is isomorphic to $\mathrm{Gr}(Hx)$, is disjoint from L (since $g\in G_L$), and is completely joined to L. If $y\in gHx$, then y is adjacent to $|Hx|-1$ vertices in gHx and $|L|$ vertices in L, and these two vertex sets are disjoint. In view of the degree of Γ, this accounts for all edges incident with y. Consequently, y can neither belong to a translate $g'Hx\neq gHx$, $g,g'\in G_L$, nor be adjacent to any vertex in such a translate. This means that any two translates g_1Hx and g_2Hx, $g_1,g_2\in G_L$, are either disjoint or identical, and if disjoint are isolated from each other.

The notation $(G_L)_{Hx}$ denotes the setwise stabilizer of the orbit Hx in G_L, i.e. $G_L\cap G_{Hx}$. Since $[G_L:(G_L)_{Hx}]\geqslant 3$, there are at least two G_L-translates of Hx outside $L\sqcup Hx=N[x]$. These translates induce clique components of $S\Gamma$ and since there are at least two such components, $S\Gamma$ must be disconnected. □

We can now give necessary and sufficient conditions for neighbor disconnection in terms of neighborhoods and orbits.

Theorem 43. *Let Γ be a vertex transitive G-graph. Then Γ is neighbor disconnected if and only if there is a subset $L\subseteq N(x)$ and a subgroup $H\leqslant G_L$ such that either*

1. $N[x]\subsetneq L\sqcup Hx\subsetneq V\Gamma$, *or*
2. $N[x]=L\sqcup Hx\subsetneq V\Gamma$ *and* $[G_L:(G_L)_{Hx}]\geqslant 3$.

Condition 2 may be replaced by

2′. $N[x]\subsetneq L\sqcup G_Lx=V\Gamma$ *and* $G_Lx\cap N[x]=Hx$ *with* $[G_L:(G_L)_{Hx}]\geqslant 3$.

If Condition 2 holds, then

$$\Gamma\cong K_m\wr E_t\wr K_n,\ \text{with}\ m\geqslant 1,\ t=[G_L:(G_L)_{Hx}],\ \text{and}\ n=|Hx|.$$

If $G=\mathrm{Aut}(\Gamma)$, then Condition 1 by itself is necessary and sufficient for neighbor disconnection.

Proof. The necessity of Condition 1 or Condition 2 and the structure of Γ if Condition 2 holds is a consequence of Lemma 39 with $L=N(U_x)$ and $H=G_{I(U_x)}$. The sufficiency of Condition 1 is Lemma 41 and the sufficiency of Condition 2 is

Lemma 42. The necessity of Condition 1 in case $G=\mathrm{Aut}(\Gamma)$ follows from Lemma 40 with $L=N(U_x)$.

Note that we do not claim that Condition 2 is equivalent to Condition 2′ but rather that ‘Condition 1 or Condition 2’ is equivalent to ‘Condition 1 or Condition 2′’. In particular, we show that Condition 2 implies either Condition 1 or Condition 2′ and that Condition 2′ implies Condition 2.

Suppose first that Condition 2 holds. Then $G_Lx\cap N[x]=G_Lx\cap(L\sqcup Hx)=Hx$. Since we already have $N[x]\subsetneq L\sqcup Hx$ and since $Hx\subseteq G_Lx$, we can have either $N[x]\subsetneq L\sqcup G_Lx\subsetneq V\Gamma$, in which case Condition 1 holds, or else $N[x]\subsetneq L\sqcup G_Lx=V\Gamma$, in which case Condition 2′ holds.

Now suppose that Condition 2′ holds. Then $N[x]=N[x]\cap V\Gamma=N[x]\cap(L\sqcup G_Lx)=L\sqcup Hx$. $L\sqcup Hx\subsetneq V\Gamma$ because $N[x]\subsetneq V\Gamma$ by the hypotheses of Condition 2′, and so Condition 2 holds. □

In view of the fact that Condition 1 is always sufficient for neighbor disconnection and is necessary when $G=\mathrm{Aut}(\Gamma)$, one might ask whether, in the case of an arbitrary G, a more perspicacious choice of L and/or H would enable us to prove that Condition 1 is always necessary. The answer is no, and the simplest counterexample is $\Gamma=K_{3,3}\cong K_2\wr E_3$. Γ is a connected neighbor disconnected vertex transitive graph with $S\Gamma=E_2$. Since $K_{3,3}\cong\mathrm{Cay}(\mathbb{Z}_6,\{1,3,5\})$, we can realize Γ as a hexagon whose vertices are labeled $0,1,2,3,4,5$ together with diagonals $\{0,3\}$, $\{1,4\}$, and $\{2,5\}$. Viewed this way, Γ has a regular action of $G=\mathbb{Z}_6$ which is just rotation of the hexagon, and of course all non-trivial subgroups of G also rotate Γ. Taking $x=0$, we have $N[0]=\{0,1,3,5\}$, $U_0=\{2,4\}$, $I(U_0)=0=K_1$, and $N(U_0)=\{1,3,5\}$.

Since $I(U_0)$ is a clique, the proof of Theorem 43 verifies that Condition 2 holds with $L=\{1,3,5\}$, $G_L=\langle 2\rangle\cong\mathbb{Z}_3$, and $H=0$. We now claim that there is no choice of L and H that would satisfy Condition 1.

Such an L must be a non-empty subset of $\{1,3,5\}$ stabilized by a subgroup of $\mathbb{Z}_6$. We have the following possibilities:

1. $L=\{1,3,5\}$ and $H=0$.
2. $L\subsetneq\{1,3,5\}$ and $H=0$ (No non-trivial subgroup of G will stabilize a proper subset of $\{1,3,5\}$ under the rotation action.)
3. $L=\{1,3,5\}$ and $H=\langle 2\rangle$.

If $H=0$, the condition $N[x]\subsetneq L\sqcup Hx$ becomes $\{0,1,3,5\}\subsetneq L\sqcup\{0\}$, which cannot be satisfied for any choice of $L\subseteq\{1,3,5\}$. This eliminates possibilities 1 and 2. In possibility 3, we have $L\sqcup Hx=V\Gamma$, violating the second proper inclusion requirement of Condition 1, and so there is no way to satisfy Condition 1.

Our final remark about Theorem 43 is that the proof also shows that if Condition 2 holds with $[G_L:(G_L)_{Hx}]=2$, then Γ still has the form $\Gamma\cong K_m\wr E_2\wr K_{|Hx|}$, but $S\Gamma\cong K_{|Hx|}$, making Γ neighbor connected.

We end this paper with a discussion of how our results improve and generalize the results of [DGS]. Let Γ, which need not be connected, be a Cayley graph for the group G. If we identify $V\Gamma$ with G, then the action of G on $V\Gamma$ is the left regular representation of G, and for any subgroup $H \leqslant G$, the orbit of g under H is a right coset Hg. We refer below to a right coset Hg as a *proper* right coset if $Hg \neq H$. Suppose that $\Gamma = \mathrm{Cay}(G,S)$, and let e be the vertex corresponding to the identity element of G. Then $S = N(e)$ and G_e is trivial. As in [DGS], we use the abbreviation S_1 for the set $N[e] = S \cup \{e\}$. Since Γ need not be connected, S need not generate G.

The translation of Proposition 38 in this context is that the survival subgraph of $\mathrm{Cay}(G,S)$ has isolated vertices if and only if G_S is non-trivial, and the number of such isolated vertices is $|G_S|$. In [DGS], a proper subset of a group is said to be *periodic* if it is a union of right cosets of a non-trivial subgroup, so Proposition 38 says that the survival subgraph of $\mathrm{Cay}(G,S)$ has isolated vertices if and only if S is periodic. This is the essential content of Theorem 2 of [DGS].

The translation, for Cayley graphs, of Theorem 43 with Condition 2′ is

Theorem 44. *Let $\Gamma = \mathrm{Cay}(G,S)$ be a Cayley graph. Then Γ is neighbor disconnected if and only if S_1 is a union of proper right cosets of some subgroup $H < G$ together with part, but not all, of H, and either*

1. *$S \cup H \neq G$, or*
2. *$S \cup H = G$, and $C = H \cap S_1$ is a subgroup of H of index at least three.*

In [DGS], a proper subset of a group is said to be *nearly periodic* if it is a union of right cosets of a non-trivial subgroup together with part, but not all, of that subgroup. With this terminology, Theorem 44 is similar to Theorem 15 of [DGS]. It is not identical for two reasons: First, Theorem 15 of [DGS] has hypotheses that exclude all Cayley graphs of the form $K_m \wr \Delta$, $m \geqslant 2$, and all group generating sets that are not unions of conjugacy classes. These restrictions eliminate cases that are handled by Theorem 44. Second, the arguments leading up to Theorem 15 in [DGS] use a dichotomy based on whether or not the survival subgraph has isolated vertices, which accounts for the presence of periodicity as well as near periodicity in the statement of Theorem 15. This dichotomy is not decisive when the restrictions imposed in [DGS] are lifted, and we see in Theorem 44 that properly qualified near periodicity conditions handle all possibilities.

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